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# An axiomatics for categories of coalgebras

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**Abstract**

We give an axiomatic account of what structure on a category $C$ and an endofunctor $H$ on $C$ yield similar structure on the category $H - Coalg$ of $H$-coalgebras. We give conditions under which completeness, cocompleteness, symmetric monoidal closed structure, local presentability, and subobject classifiers lift. Our proof of the latter uses a general result about the existence of a subobject classifier in a category containing a small dense subcategory. Our leading example has $C = Set$ with $H$ the endofunctor for which a coalgebra is a finitely branching (labelled) transition system. We explain that example in detail.

## 1 ntrodu■tion

Given an endofunctor $H$ on the category $Set$, an *H-coalgebra* is a set $X$ together with a function $x : X \longrightarrow HX$. A leading example of such an $H$ is given by the functor $P_\omega$ that takes a set $X$ to the set of finite subsets of $X$, with the behaviour of $H$ on maps given by direct image. An $H$-coalgebra is then a finitely branching transition system. A variant, is given by starting with a set $L$ and letting $HX$ be the set of finite subsets of $L \times X$. For that $H$, an $H$-coalgebra is a finitely branching labelled transition system, with labels in $L$. Many other $H$'s have been investigated too: for a detailed introduction

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and further examples, see Jacobs and Rutten's tutorial [4] and the papers cited therein, especially Rutten's [11].

Much of the theory of coalgebras has been directed towards an algebraic account of coinduction in computer science. One can account for bisimulation and coinductive definitions of data types in terms of coalgebras and maps of coalgebras (see [4] and the papers cited therein). For the two $H$'s cited above, the maps in $H - Coalg$ amount exactly to the usual notion of functional bisimulation of transition systems. So it seems natural to ask, in general, what is the structure of $H - Coalg$? For instance, is it complete, cocomplete, or symmetric monoidal closed? If not, then under what conditions is it thus? In this paper, we give conditions under which it has all that structure, and more. In particular, under a condition, it has a subobject classifier. That is significant in that, for a particular $H$, namely that $H$ taking a set $X$ to the set of its nonempty finite subsets, the subobject classifier amounts to the set of hypersets [12], or the set of sets satisfying Aczel's anti-foundation axiom [1]. For that specific $H$, the category of $H$-coalgebras is studied in detail in [13,15] and in Watanabe's thesis [16].

There is a rapidly growing body of research on the category $H - Coalg$. One substantial work is Michael Barr's paper [2], in which he showed that the forgetful functor $U : H - Coalg \longrightarrow Set$ has a right adjoint, and analysed structures relevant to that. He also related that result to Aczel's non-well founded set theory. Despite restricting his attention to $Set$, Barr's proof was axiomatic; but he did not extend his result to a general analysis of the structure of $H - Coalg$. Another article, to appear in this volume, is by James Worrell [17], who addresses the same topic as we do, but with a somewhat different emphasis. Together with Peter Johnstone, the work of our two papers is currently being combined and extended [5]. Finally, Rutten's technical report [11] overlaps a little with our work: a few basic results agree, but his emphasis is on functors that preserve weak pullbacks, a condition we do not consider.

Here, our approach is axiomatic. What we mean by that is that we do not restrict attention to $Set$ as a base category, and we do not prove results about a specific endofunctor, although all our results apply to a large class of endofunctors. What we do is consider a base category $C$ with some structure, for instance that of a symmetric monoidal closed category, and an arbitrary endofunctor $H$ on it satisfying some conditions; then prove that $H - Coalg$ has the structure we assert. The category $Set$ has all the structure we consider, as does any Grothendieck topos. All the structures and properties we consider on endofunctors are mild and hold of our leading examples.

The abstract category theory underlying this paper is largely based on Makkai and Pare's accessible categories [10]. That work is not central although helpful for one of our major results, that in which we assert that $H - Coalg$ has a subobject classifier, but it is central to most others.

Once one has an account of $H - Coalg$ and hence an algebraic account of categories in which maps are functional bisimulations, an immediate following question is about categories of bisimulations. That is future work, but observe that for any category $E$ with pullbacks, one can consider the bicategory $Span(E)$, whose objects are those of $E$ and whose 1-cells are given by spans of arrows in $E$. So, in this case, an object would be given by an $H$-coalgebra, and a map from $A$ to $B$ would be a pair of functional bisimulations from an $H$-coalgebra $D$ into $A$ and $B$. That is exactly the way Joyal et al defined bisimulation in their study of open maps for bisimulation [6], i.e., they gave a notion of functional bisimulation, then said a bisimulation is a span of functional bisimulations between them. So we leave a study of a bicategory of bisimulations for future work, but expect it to involve a study of $Span(E)$ or possibly $Rel(E)$, or a variant, where $E = H - Coalg$.

This paper is organized as follows. In Section 2, we give conditions under which $H - Coalg$ is cocomplete and has a symmetric monoidal structure. In Section 3, we explain the notion of accessible category, and give a condition under which $H-Coalg$ is accessible, and hence has a small dense subcategory. That is the heart of our use of Makkai and Pare's work on accessible categories, and it appears in Barr's paper [2]. In the presence of colimits, it follows that $H-Coalg$ is locally presentable. Accessibility allows us to deduce immediately that $H - Coalg$ is complete, that the symmetric monoidal structure is closed, and that the forgetful functor to the base category has a right adjoint, as in Barr's paper [2]. Then, in Section 4, we give a condition under which $H-Coalg$ has a subobject classifier. An accessible category always has a small dense subcategory, as we shall explain. Here, we use density to give a general result that a cocomplete category with a small dense subcategory, hence for instance every locally presentable category, has a subobject classifier if there is an object that classifies subobjects of objects of the dense subcategory. Then, armed with that result, we give our proof that $H - Coalg$ has a subobject classifier. In fact, we prove a more general result to the effect that any category that contains a small dense subcategory and has a functor into an elementary topos satisfying a few conditions has a subobject classifier, and deduce our result about $H - Coalg$ from that. Finally, in Section 5, we investigate one of our leading examples in detail.

## 2 outine results

In this section, we give a few routine results about $H - Coalg$. We include them largely for completeness, as we shall need them later.

**Theorem 2.1** *If $C$ is cocomplete and $H$ is any endofunctor on $C$, then $H - Coalg$ is cocomplete and the forgetful functor $U : H - Coalg \longrightarrow C$ preserves colimits.*

The proof is a routine calculation, using the definition of colimits (see [11] Thm 4.5 for essentially the same result). It is also routine to verify

**Proposition 2.2** *The forgetful functor $U : H - Coalg \longrightarrow C$ reflects isomorphisms, i.e., if $f : A \longrightarrow B$ is a map in $H - Coalg$ for which $Uf : UA \longrightarrow UB$ is an isomorphism, then $f$ is an isomorphism.*

These two results imply most of [11] Prop 4.7, i.e., that epimorphisms in $H - Coalg$ are those maps sent by $U$ to epimorphisms in $C$, and that monomorphisms are reflected by $U$. The rest of Rutten's result assumes the preservation of weak pullbacks by $H$, which we do not assume, but see the proof of Corollary 4.6.

Of greater interest here, a symmetric monoidal endofunctor on a symmetric monoidal category $C$ consists of an endofunctor $H : C \longrightarrow C$ together with two natural transformations, with components $\bar{H}_{(X,Y)} : HX \otimes HY \longrightarrow H(X \otimes Y)$ and $\hat{H} : I \longrightarrow HI$ subject to four coherence axioms to the effect that these natural transformations respect the coherence isomorphisms of the symmetric monoidal structure. Often we write $H$ for a symmetric monoidal endofunctor, leaving the rest of the structure implicit. An endofunctor may have more than one symmetric monoidal structure on it.

**Example 2.3** The endofunctor $P_\omega$ on $Set$ has two symmetric monoidal structures, one given by the map $\bar{P}_\omega : P_\omega X \times P_\omega Y \longrightarrow P_\omega(X \times Y)$ sending $(A, B)$ to $A \times B$, with unit given by sending 1 to $\{1\}$, and the other given by sending $(A, B)$ to $\{(x, y) : x \epsilon A \vee y \epsilon B\}$ with the unit given by sending 1 to the empty set. The former is the one of primary interest, as it corresponds to synchronization. For the endofunctor $P_\omega(L \times -)$ for finite $L$, we can give a symmetric monoidal structure by the map $\bar{P}_\omega(L \times -) : P_\omega(L \times X) \times P_\omega(L \times Y) \longrightarrow P_\omega(L \times X \times Y)$ sending $(A, B)$ to $\{(l, a, b) : \ (l, a) \epsilon A \wedge (l, b) \epsilon B\}$, with the unit by sending 1 to $\{(l, 1) : l \epsilon L\}$.

**Theorem 2.4** *Let $C$ be a symmetric monoidal category and let $H$ be a symmetric monoidal endofunctor on $C$. Then $H - Coalg$ has a symmetric monoidal structure that is preserved strictly by the forgetful functor $U : H - Coalg \longrightarrow C$.*

**Proof.** Given $H$-coalgebras $(X, x)$ and $(Y, y)$, define $(X, x) \otimes (Y, y)$ to have object $X \otimes Y$, the tensor product in $C$, with the map from $X \otimes Y$ to $H(X \otimes Y)$ given by composing $x \otimes y$ with $\bar{H}_{(X,Y)}$ It is routine to verify that $H - Coalg$ is symmetric monoidal, using the axioms on $H$ and those of the symmetric monoidal structure of $C$. Moreover, by construction of the tensor product, $U$ preserves it strictly. □

It is routine to verify that the various examples of $H$ that most interest us satisfy the condition of the Theorem.

## 3 Accessibility

In this section, we shall give a condition on a category $C$ and on an endofunctor $H$ that forces $H - Coalg$ to be what is called an accessible category, and hence have what is known as a small dense subcategory.

One of the reasons that accessibility of a category is of fundamental importance is as follows. If one has a preordered set, then it has all infima if and only if it has all suprema. So, since completeness of a category extends the notion of a preorder having all infima, and since cocompleteness of a category extends the notion of a preorder having all suprema, it is natural to ask whether a category is complete if and only if it is cocomplete. But that is not the case in general for large categories, and the most interesting categories such as *Set* are large. If one follows the argument for preorders, the point where it falls down for categories is a size question: when one says that a category is complete, one means that every small diagram has a limit, and dually for cocompleteness. But in generalising the argument for preorders, at one point one needs a colimit of a large diagram. For instance, the terminal object of a category $C$, if it exists, is a colimit of the identity functor on $C$: but the domain of the diagram giving that colimit is $C$, which is typically a large category. There is an account of this issue in Mac Lane's book [9]. So one might ask under what condition on a category does it follow that the category is complete if and only if it is cocomplete? A related question is, given a functor that preserves all colimits, under what condition does it have a right adjoint? One particularly natural condition on a category that allows such results is accessibility. The basic reference for accessible categories is [10].

**Definition 3.1** Let $\kappa$ be an infinite regular cardinal (such as $\omega$). Then, a small category $I$ is *$\kappa$-filtered* if for any category $J$ of cardinality less than $\kappa$, any diagram $D : J \longrightarrow I$ has a cocone over it. A colimit is *$\kappa$-filtered* when it is a colimit of a diagram whose domain is $\kappa$-filtered.

**Definition 3.2** An object $X$ of a category $C$ is called $\kappa$-presentable if the homfunctor $C(X, -) : C \longrightarrow Set$ preserves $\kappa$-filtered colimits.

For example, taking $\kappa = \omega$, a set is $\kappa$-presentable if and only if it is finite. More generally, for arbitrary $\kappa$, a set is $\kappa$-presentable if and only if it has cardinality less than $\kappa$.

**Definition 3.3** A category $C$ is $\kappa$-accessible if

(i) $C$ has $\kappa$-filtered colimits, and

(ii) there is a small full subcategory $B$ of $C$ consisting of $\kappa$-presentable objects, such that every object of $C$ is a $\kappa$-filtered colimit of a diagram that factors through $B$.

A category is *accessible* if it is $\kappa$-accessible for some infinite regular cardinal $\kappa$.

For instance, *Set* is $\omega$-accessible, because every set is expressible as the union of its finite subsets. Similarly, *Cat* is $\omega$-accessible because every small category is a filtered colimit of finitely presentable categories. In general, any locally presentable category is accessible, as are many other categories.

A consequence of category being accessible is that it has what is called a small dense subcategory (see [10] Prop 2.1.5). That is not a difficult result, but it is convenient, in that it gives a canonical description of each object of the category as a colimit of a diagram factoring through a small subcategory. Specifically,

**Definition 3.4** A small full subcategory $T$ is *dense* in $C$ with inclusion $J : T \longrightarrow C$ if every object $X$ of $C$ is a colimit of the diagram

$$J(-)/X \longrightarrow T \longrightarrow C$$

where $J(-)/X$ is the comma category, i.e., an object consists of an object $t$ of $T$ together with a map from $Jt$ into $X$, and an arrow is an arrow in $T$ making the diagram commute; with the functors given by projection $\pi : J(-)/X \longrightarrow T$ and the inclusion $J : T \longrightarrow C$.

There are various characterisations of the notion of density, and there are a few mild variants, such as not insisting that $J$ be full, or not insisting that $T$ be a subcategory of $C$; but they all amount to the same idea. An elegant and useful characterisation of density is

**Proposition 3.5 ([7])** *$T$ is dense in $C$ if and only the functor from $C$ to $[T^{op}, Set]$ sending $X$ to $C(J(-), X) : T^{op} \longrightarrow Set$ is fully faithful.*

The proof is straightforward. For much of our analysis, we shall use the existence of a small dense subcategory as a standard assumption: it is a little weaker than the assumption of accessibility, and the way we obtain $H-Coalg$ having a small dense subcategory is always via an accessibility condition and argument. But some of our results only require density, and it is convenient in that it gives a canonical colimit for each object.

Returning to accessibility and how we obtain $H - Coalg$ as an accessible category

**Definition 3.6** A functor between $\kappa$-accessible categories is $\kappa$-accessible if it preserves $\kappa$-filtered colimits. A functor is accessible if it is $\kappa$-accessible for some $\kappa$.

All of the endofunctors on $Set$ of interest to us are accessible. See Barr's paper [2] for an account. The central point about $\kappa$-filtered colimits that make them easy to handle is that 1 is $\kappa$-presentable, and so an element of a $\kappa$-filtered colimit is always the image of an element of one of its components, and two elements are equal in the colimit if and only if they are equal in some component. This contrasts with coequalizers. For an example of an accessible endofunctor

**Example 3.7** Let $P_\omega$ denote the endofunctor on $Set$ that sends a set $X$ to the set of finite subsets of $X$. Then $P_\omega$ preserves $\omega$-filtered colimits: let $X$ be an $\omega$-filtered colimit of $X_i$. Then, given a finite subset $F$ of $X$, each element of $F$ must lie in the image of some $X_i$. By filteredness, it follows that there is some $X_k$ for which all elements of $F$ are in the image of $X_k$. Moreover, any two finite subsets of $X$ are equal if and only if they may be shown to be equal in some $X_k$. That completes the proof.

One of the central theorems, Theorem 5.1.6, of [10] yields

**Theorem 3.8** *For any accessible category $C$ and any accessible endofunctor $H$ on $C$, the category $H - Coalg$ is accessible.*

**Proof.** First observe that $H - Coalg$ has a limit-like universal property in $CAT$, namely as the universal diagram in $CAT$ of the form

$$\begin{tikzcd} & C \arrow[dd, "H"] \\ D \arrow[ur, "U"] \arrow[dr, "U"'] \arrow[r, phantom, "\Uparrow\alpha"] & \\ & C \end{tikzcd}$$

where $\alpha$ is a natural transformation, i.e., in the diagram, we may replace $D$ by $H - Coalg$, $U$ by the forgetful functor $U_H : H - Coalg \longrightarrow C$, and $\alpha$ by the natural transformation $\gamma : U_H \Rightarrow HU_H$ with $(X, x)$-component $x : X \longrightarrow HX$; and for any other diagram in $CAT$ of this form, there is a unique functor $Q : D \longrightarrow H - Coalg$ making $U = U_H Q$ and $\gamma Q = \alpha$.

Now, accessible categories are characterized, generalising Gabriel Ulmer duality, by the fact that they are the categories of models for sketches. So, if one passes along the duality between the category of accessible categories and that of sketches, one may replace $C$ by the corresponding sketch $Sk(C)$, and replace $H$ by the corresponding map of sketches $Sk(H)$. This being a duality, limit-like properties of the category of accessible categories correspond to colimit-like properties of the category of sketches. But the category of sketches is cocomplete. So if we take the colimit-like universal diagram in

*Sketch*

$$\begin{tikzcd} & Sk(C) \arrow[dl, "V"'] \\ E \arrow[r, phantom, "\Uparrow\beta"] & \\ & Sk(C) \arrow[ul, "V"] \arrow[uu, "Sk(H)"'] \end{tikzcd}$$

and pass back along the duality, we obtain a limit construction in the category of accessible categories. One can routinely check that that diagram satisfies the defining limiting property of $H - Coalg$, so is isomorphic to $H - Coalg$. Thus $H - Coalg$ is accessible. □

For more detail of the proof, see [10]. For our leading example, we shall give an independent proof that $H - Coalg$ has a small dense subcategory: that is a weaker condition than accessibility, but it suffices for our results here. In the case that $HX$ is the set of finite subsets of $L \times X$ for a finite set $L$, the category of finitely branching $L$-labelled trees gives a small dense subcategory.

The result that $H - Coalg$ is accessible if $C$ and $H$ are accessible is fundamental for us. We have already seen that if $C$ is cocomplete, then $H - Coalg$ is cocomplete. Several equivalent definitions of locally presentable category were given by Gabriel and Ulmer [3]; one of them, in our terminology, was

**Definition 3.9** A *locally presentable category* is a cocomplete accessible category.

There has been considerable study of locally presentable categories, for instance in [3] and [8], and the class of locally presentable categories is of central importance to category theory. The accessibility result immediately yields

**Corollary 3.10** *If $C$ is locally presentable and $H$ is accessible, then $H - Coalg$ is locally presentable.*

A fundamental result about locally presentable categories is an immediate corollary of the following [8]

**Theorem 3.11** *Let $C$ be cocomplete and have a small dense subcategory, and let $D$ be cocomplete. Then any colimit preserving functor $F : C \longrightarrow D$ has a right adjoint.*

**Proof.** Let $T$ be dense in $C$, and given $X$ in $D$, consider $colim(Fj(-)/X \longrightarrow T \longrightarrow C)$, where the first component is given by projection, and the second is the inclusion $j : T \longrightarrow C$. Now follow the usual argument for preorders: that argument now works because the colimit is taken over a small diagram, and therefore exists in $C$. $\square$

**Corollary 3.12** *If $C$ is cocomplete with a small dense subcategory, then $C$ is complete.*

**Proof.** The diagonal functor $\Delta : C \longrightarrow [I, C]$ preserves colimits for any small category $I$. So it has a right adjoint. $\square$

**Corollary 3.13** *If $C$ is locally presentable and $H$ is accessible, then $H{-}Coalg$ is complete.*

This result may seem surprising. Note carefully what it does say and what it does not say. It implies, for instance, the existence of all binary products in the category $Trans_L$. But it does not imply that the product is a simple construction: in particular, it does not imply that it is the product in the category of transition systems with the usual maps of transition systems.

**Corollary 3.14 ([2])** *If $C$ is locally presentable and $H$ is accessible, the forgetful functor $U : H - Coalg \longrightarrow C$ has a right adjoint. Moreover, the right adjoint is accessible.*

**Proof.** The existence of a right adjoint follows because $U$ preserves colimits. Its accessibility follows by its construction. $\square$

**Corollary 3.15** *If $C$ is locally presentable and $H$ is accessible, the forgetful functor $U : H - Coalg \longrightarrow C$ is comonadic, expressing $H - Coalg$ as the category of coalgebras for an accessible comonad on $C$.*

**Proof.** This is a routine argument using the dual of Beck's monadicity theorem (see [9]). One needs to show that $U$ preserves the equalizers of $U$-split equalizers, but that follows directly from the definitions of $H$-coalgebra and $U$-split equalizer as in [9]. $\square$

**Corollary 3.16** *If $C$ is locally presentable and symmetric monoidal closed, and if $H$ is symmetric monoidal and accessible, then $H - Coalg$ is symmetric monoidal closed.*

**Proof.** $H - Coalg$ is cocomplete, with $U$ preserving colimits, and it is symmetric monoidal, with $U$ preserving the symmetric monoidal structure; recall also that $U$ reflects isomorphisms. Let $(X, x)$ be any object in $H - Coalg$. We need to show that $(X, x) \otimes - : H - Coalg \longrightarrow H - Coalg$ preserves colimits. But that follows by routine argument based on the above results, and the fact that, since $C$ is closed, $X \otimes - : C \longrightarrow C$ preserves colimits. $\square$

**Example 3.17** All the symmetric monoidal structures given in Example 2.3 turn out to be closed by Corollary 3.16.

Finally, we can make one more observation putting the above together.

**Definition 3.18** A symmetric monoidal closed category is called *locally presentable as a closed category* [8] if it is locally presentable and if the $\kappa$-presentable objects are closed under $\otimes$ and $I$ for some $\kappa$.

**Corollary 3.19** *If $C$ is locally presentable as a closed category and if $H$ is symmetric monoidal and accessible, then $H - Coalg$ is locally presentable as a closed category.*

**Proof.** By the results above, we need only to show that the $\kappa$-presentable objects of $H - Coalg$ include the unit and are closed under $\otimes$ for some $\kappa$. Recall that the right adjoint $G$ of $U$ is accessible. So it preserves $\kappa$-filtered colimits for some $\kappa$. Taking that $\kappa$, it follows that if $(X, x)$ is $\kappa$-presentable in $H - Coalg$, then $X$ is $\kappa$-presentable in $C$. From this point, the rest follows routinely from the definitions of $\kappa$-presentable object and $\kappa$-filtered colimit.$\Box$

This result is significant because the deeper results of the theory of enriched categories are based upon enrichment in a category that is locally presentable as a closed category (see [8]). So this tells us that, for all endofunctors of interest to us, the category $H - Coalg$ is a suitable basis for enriched category theory. So for instance, the category of finitely branching (labelled) transition systems and functional bisimulations is suitable for enriched category theory, so one may reasonably speak of a hom possessing the structure of a transition system. This could potentially be of considerable interest in modelling dynamic properties of programs.

## 4 he suojent alssifier

In this section, we shall take as a basic assumption that we consider a category $D$ (for which our leading example is any category of the form $H - Coalg$ for accessible $H$ on locally presentable $C$) that is cocomplete and has a small dense subcategory $T$. By Corollary 3.12, it follows that $D$ is complete. We shall consider conditions under which $D$ has a subobject classifier.

Our first result is about size. The condition that a category $D$ has a subobject classifier is the statement that the functor $sub : D^{op} \longrightarrow Set$ taking an object $X$ to its set of subobjects (assuming it has such a small set) with behaviour on maps given by pullback (again assuming such exist), is representable. Any category $D$ which is cocomplete and has a small dense subcategory $T$ does admit the existence of the functor $sub$ since $D$ is complete and is a full subcategory of $[T^{op}, Set]$. But the representability is a condition that involves all objects of $D$ rather than a small family of them. So we need to cut that down to a condition about a small family. In fact, we can prove

**Theorem 4.1** *Let $D$ be cocomplete with a small dense subcategory $T$ with inclusion $J : T \longrightarrow D$. Suppose there exists an object $\Omega$ and a map $true : 1 \longrightarrow \Omega$ in $D$ such that pulling back along $true$ yields an isomorphism from the functor $D(J(-), \Omega) : T^{op} \longrightarrow Set$ to the functor $subJ(-) : T^{op} \longrightarrow Set$.*

*Then $\Omega$ together with true is a subobject classifier for $D$.*

**Proof.** Given a monomorphism $j : Y \longrightarrow X$ in $D$, consider the expression of $X$ as the colimit

$$colim(J(-)/X \longrightarrow T \longrightarrow D).$$

and take the pullback of $Y$ along each of the coprojections

$$\begin{tikzcd} t \wedge Y \arrow[r] \arrow[d] & Y \arrow[d, "j"] \\ t \arrow[r] & X \end{tikzcd}$$

A pullback of a monomorphism is a monomorphism, and so this determines a map from $t$ to $\Omega$ for each map from $t$ to $X$. These maps form a cocone, and hence yield a map from $X$ to $\Omega$ which we call $\chi_Y$. It remains to show that

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "j"'] & 1 \arrow[d, "true"] \\ X \arrow[r, "\chi_Y"'] & \Omega \end{tikzcd}$$

commutes, and is a pullback in $D$. To see that it commutes, consider $Y$ as a canonical colimit. Every map from a $t$ to $Y$ composes to give a map into $X$, and since the composite factors through the monomorphism $j : Y \longrightarrow X$, the pullback of $j$ along it is the identity on $t$. Now, by routine manipulation of pullbacks and colimits, we have the commutativity.

To see that it is a pullback, by density it suffices to show that for every $t$ in $T$ and every map $f : t \longrightarrow X$ making the evident square commute, $f$ factors through $j : Y \longrightarrow X$. Since *true* is a monomorphism, by the commutativity, the identity on $t$ is the pullback of *true* along $\chi_Y.f : t \longrightarrow \Omega$. But taking the pullback of $j$ along $f$, then taking the corresponding map from $t$ into $\Omega$ is $\chi_Y.f$ by construction of $\chi_Y$. So the pullback of $j$ along $f$ is the identity, and so $f$ factors through $j$.

The unicity of $\chi_Y$ is routine to verify, using the unicity part of the property of $X$ as a colimit. □

With a little effort, one can calculate what the terminal object in a cocomplete category with a small dense subcategory must be, as a colimit; and one can do likewise with a subobject classifier, if one exists. We want expressions as colimits because in our leading examples, those categories of the form $H - Coalg$ for an accessible functor $H$ on a locally presentable category $C$, we know how to calculate colimits, as they are given as colimits in $C$, whereas we do not have a simple description of limits.

We must first discuss a special class of colimits called coends.

**Definition 4.2** Given a small category $I$ and a functor $F : I^{op} \times I \longrightarrow D$, a *coend* of $F$, denoted $\int^i F(i,i)$, is a coequalizer of the coproduct $\sum_i F(i,i)$ given by coequalizing the evident two maps from $\sum_{f:i \to j} F(j,i)$ to $\sum_i F(i,i)$.

The coend of primary interest to us has $I = T$ and $F : T^{op} \times T \longrightarrow D$ given by $F(s,t) = \sum_{sub(s)} t$, i.e., a coproduct of $sub(s)$ copies of $t$. There is a general theory of coends (see for instance [7]). In particular, for any object $A$ of a cocomplete category $D$ with a small dense subcategory $T$, one has

$$\int^t \sum_{D(Jt,A)} t = A.$$

With this definition, and with some calculation, one can conclude

**Proposition 4.3** *Let $D$ be cocomplete with a small dense subcategory $J : T \longrightarrow D$. Then*

(i) *the terminal object in $D$ is $colim(J : T \longrightarrow D)$*

(ii) *if $D$ has a subobject classifier $\Omega$, then it must be the coend $\int^t \sum_{sub(t)} t$, with $true : 1 \longrightarrow \Omega$ given by the cocone determined by factoring through the coprojection of $t$ into the $id_t$-component of $t$, and given a monomorphism $j : Y \longrightarrow t$, with the map $\chi_Y : t \longrightarrow \Omega$ given by factoring through the coprojection of $t$ into the $Y$-component of $t$.*

**Proof.** If $D$ has a subobject classifier $\Omega$, then $sub(t)$ is isomorphic to $D(t, \Omega)$, and hence the above colimit defines $\Omega$. □

We shall now use Theorem 4.1 to give a condition under which a category $D$ has a subobject classifier when a related category $C$ has one. First, we need a lemma. Say that a functor $U : D \longrightarrow C$ *weakly preserves* a pullback if it sends the pullback to a weak pullback, where a *weak pullback* satisfies the existence but not unicity part of the definition of pullback.

**Lemma 4.4** *Let $U : D \longrightarrow C$ weakly preserve pullbacks of monomorphisms. Then $U$ preserves pullbacks of monomorphisms, so in particular, preserves monomorphisms.*

**Proof.** Given a monomorphism $m$ in $D$, the pullback of $m$ along itself is the identity. By weak preservation of pullbacks, it follows that $Um$ is a monomorphism. An arbitrary pullback of a monomorphism in $D$ must be a monomorphism, so be sent by $U$ to a monomorphism in $C$. Together with weak preservation of pullbacks of monomorphisms, that implies that $U$ preserves the pullback. □

**Theorem 4.5** *Let $D$ be cocomplete with a small dense subcategory $J : T \longrightarrow D$. Let $C$ be an elementary topos, and suppose $U : D \longrightarrow C$ has a right adjoint, reflects isomorphisms, and weakly preserves pullbacks of monomorphisms. Then $D$ has a subobject classifier.*

Note that we do not assume that $U$ preserves all finite limits. In particular, $U$ does not preserve the terminal object in many of our leading examples.

**Proof.** It suffices to prove that $D$ has an object $\Omega$ that classifies subobjects of objects $t$ of the small dense subcategory $T$. So, given a monomorphism $j : Y \longrightarrow t$ in $D$, consider the commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "j"'] & 1 \arrow[d, "true"] \\ X \arrow[r, "\chi_Y"'] & \Omega \end{tikzcd} \qquad (4.1)$$

where $\Omega$, $true$, and $\chi_Y$ are defined as in the proposition.

Also consider the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & 1 \arrow[d, "true"] \\ X \arrow[r, "\chi_Y"'] & \Omega \end{tikzcd} \qquad (4.2)$$

By definition, there is a unique comparison map $c : Y \longrightarrow P$ making the triangle into $X$ commute. We seek to show that $c$ is an isomorphism.

Observe that there is a map $\gamma : U(\Omega) \longrightarrow \Omega_C$ determined by the forgetful function $sub_D(t) \longrightarrow sub_C(Ut)$ for each $t$: that this map is defined uses the assumptions that $U$ has a right adjoint, thus preserves the colimit, and, by the Lemma, that $U$ preserves monomorphisms and pullbacks of monomorphisms, the latter being needed to prove the naturality in $t$.

It is routine to verify that the square

$$\begin{tikzcd} U(1_D) \arrow[r] \arrow[d, "U(true)"'] & 1_C \arrow[d, "true"] \\ U\Omega_D \arrow[r, "\gamma"'] & \Omega_C \end{tikzcd}$$

and the triangle

$$\begin{tikzcd} UX \arrow[r, "U\chi_Y"] \arrow[dr, "\chi_{UY}"'] & U\Omega_D \arrow[d, "\gamma"] \\ & \Omega_C \end{tikzcd}$$

commute. Thus, the diagram given by applying $U$ to (4.1) and composing with $\gamma$ is a pullback. Of course, applying $U$ to (4.2) and composing with $\gamma$ gives a commutative square. Consequently, we have a map in $C$ from $UP$ to $UY$ that commutes with the two maps into $UX$. Those two maps into $UX$ are both monomorphisms, and so $Uc$ is invertible. Hence, since $U$ reflects isomorphisms, $c$ is invertible. □

**Corollary 4.6** *Let $H$ be an accessible endofunctor on a locally presentable elementary topos $C$, and suppose that $H$ weakly preserves pullbacks. Then $H - Coalg$ has a subobject classifier.*

**Proof.** We need to show that if $H$ weakly preserves pullbacks, so does $U : H - Coalg \longrightarrow C$. This is a routine argument: given a diagram in $H - Coalg$, take the pullback in $C$, and use weak preservation by $H$ to lift the pullback to a commutative square in $H - Coalg$. Now use the pullback property in $H - Coalg$ and in $C$ to show that the pullback in $C$ splits that in $H - Coalg$. □

This proof is essentially [11] Thm 4.6 and gives a slightly stronger formulation of the last part of [11] Prop 4.7. Of course, this result does not say that $H - Coalg$ need be a topos: see [5] for a counterexample. As we mentioned above, the comonad induced by $H$ does not in general preserve finite limits. Moreover, if $H$ was symmetric monoidal, we obtained a symmetric monoidal closed structure on $H - Coalg$, not a cartesian closed structure. However, we did use the fact that $C$ was a topos in the proof when we gave the map $\gamma$ into the subobject classifier of $C$. See [5] for a more extensive analysis of this result.

The result has several consequences given by routinely following the algebraic theory of toposes but replacing the cartesian closed structure by symmetric monoidal closed structure. For instance,

**Corollary 4.7** *If $H$ is accessible and symmetric monoidal, and $H$ weakly preserves pullbacks, with $C$ locally presentable as a closed category and containing a subobject classifier (e.g., if $C$ is a Grothendieck topos),*

(i) *every monomorphism in $H - Coalg$ is an equalizer.*

(ii) *the functor from $H - Coalg^{op}$ to $H - Coalg$ sending an object $X$ to $[X, \Omega]$ is monadic.*

(iii) *every epimorphism in $H - Coalg$ is a coequalizer.*

This completes our general axiomatic development of what structures and properties on a category $C$ and an endofunctor $H$ on $C$ give rise to corresponding structures and properties on the category of $H$-coalgebras. This analysis relates closely to that of [17]; see [5] for an extension of [17] and this paper.

## 5 elled trnsition systems

In this final section of the paper, we discuss a particular example of an endofunctor $H$ on $Set$ in detail. Consider the endofunctor $P_\omega(L \times -)$ on $Set$ that takes a set $X$ to the set of finite subsets of $L \times X$. The category of coalgebras amounts to the category whose objects are finitely branching $L$-labelled transition systems, with labels in a set $L$; we denote it by $Trans_L$.

We show that $Trans_L$ is complete and cocomplete with a subobject classifier. Cocompleteness follows from Theorem 2.1. As for completeness, we can see it from the existence of a small dense subcategory in $Trans_L$. The existence of a small dense subcategory can be shown by accessibility of $P_\omega(L \times -)$ in the same way as for $P_\omega$ in Example 3.7, but we can construct a small dense subcategory explicitly as follows.

**Definition 5.1** Let $\mathbb{N}$ be the set of natural numbers and $(L \times \mathbb{N})^*$ be the set of finite words over $L \times \mathbb{N}$. Let $A$ be a subset of $(L \times \mathbb{N})^*$. We say

- $A$ is *prefix closed* if $w \in A$ implies $v \in A$ for all prefixes $v$ of $w$.
- $A$ is *locally finite* if for each $v \in A$ the set $\{x \in L \times \mathbb{N} : \ v.x \in A\}$ is a finite set of the form $\{(l_1, 1), (l_2, 2), \ldots, (l_{n_v}, n_v)\}$ with $l_i \in L$ for each $1 \leq i \leq n_v$.

Observe that every prefix closed subset $A \subset (L \times \mathbb{N})^*$ contains the empty word $\epsilon$. Each prefix closed locally finite subset $A \subset (L \times \mathbb{N})^*$ determines a $P_\omega(L \times -)$-coalgebra $(A, \tau_A)$ by $\tau_A(w) = \{(l, w.(l,i)) : \ w.(l,i) \epsilon A\} \subset L \times A$: we call it a *finitely branching labelled tree*. Finitely branching labelled trees and morphisms of $P_\omega(L \times -)$-coalgebras define a category that we denote by $Tree_L$. Hence $Tree_L$ is a full subcategory of $Trans_L$: we denote the inclusion by $i : Tree_L \longrightarrow Trans_L$. By construction, the category $Tree_L$ is small.

Let $(D, d)$ be a $P_\omega(L \times -)$-coalgebra. Define a *numbering* $\alpha$ on $(D, d)$ as follows. For each $x \epsilon D$, number the elements of $d(x)$ from 1 to $|d(x)|$, and define the function $\alpha_x : d(x) \longrightarrow L \times \mathbb{N}$ by $\alpha_x(z) = (l, n)$, where $l$ is the label of $z \epsilon d(x)$ and $n$ is the number of $z$ in $d(x)$. For each $x \epsilon D$, we call a finite sequence $z_1, z_2, \ldots, z_n$ of $L \times D$ a *path* from $x$ to $\partial_2(z_n) \epsilon D$ if $z_1 \epsilon d(x)$ and $z_{k+1} \epsilon d(\partial_2(z_k))$ for each $1 \leq k \leq n-1$, where $z = (\partial_1(z), \partial_2(z))$. Given a numbering $\alpha$ on $(D, d)$ and $x \epsilon D$, define $\mathrm{Path}_\alpha(x) \subset (L \times \mathbb{N})^*$ to be

$$\{\epsilon\} \cup \{\alpha_x(z_1).\alpha_{\partial_2(z_1)}(z_2) \ldots \alpha_{\partial_2(z_{n-1})}(z_n) : \ z_1, z_2, \ldots, z_n \text{ is a path from } x \text{ in } D\}.$$

Then $\mathrm{Path}_\alpha(x)$ is a prefix closed locally finite subset of $(L \times \mathbb{N})^*$. Hence it determines an object of $Tree_L$ that we also denote by $\mathrm{Path}_\alpha(x)$. There is a canonical arrow $\gamma_x : i(\mathrm{Path}_\alpha(x)) \longrightarrow (D, d)$ in $Trans_L$ defined inductively by

$\gamma_x(\epsilon) = x$, and for $w.(l,n)\epsilon\mathrm{Path}_\alpha(x)$ with $(l,n)\epsilon L \times \mathbb{N}$, $\gamma_x(w.(l,n)) = y$ with $\alpha_{\gamma_x(w)}((l,y)) = (l,n)$.

**Lemma 5.2** *Let $D$ be an object of $Trans_L$ and let $\alpha$ be a numbering on it. For each object $A$ of $Tree_L$ and each arrow $f : i(A) \longrightarrow D$ in $Trans_L$, there exists $x\epsilon D$ and an arrow $\bar{f} : A \longrightarrow \mathrm{Path}_\alpha(x)$ such that the following diagram commutes.*

$$\begin{tikzcd} i(A) \arrow[r, "i(\bar{f})"] \arrow[dr, "f"'] & i(\mathrm{Path}_\alpha(x)) \arrow[d, "\gamma_x"] \\ & D \end{tikzcd}$$

Now let $\Gamma(D)$ be the free category generated by the graph $(D,d)$ and define the functor $E : \Gamma(D)^{op} \longrightarrow Tree_L$ by

- on objects, $E(x) = \mathrm{Path}_\alpha(x)$,
- an edge of $\Gamma(D)$ amounts to a pair $(x,z)$ with $x\epsilon D$ and $z\epsilon d(x)$. So define $E$ on arrows by defining $E((x,z)) : \mathrm{Path}_\alpha(\partial_2(z)) \longrightarrow \mathrm{Path}_\alpha(x)$ by

$$E((x,z))(w) = \alpha_x(z).w.$$

**Lemma 5.3**

$$colim(i \circ E) \cong (D,d)$$

**Proposition 5.4** *The category $Tree_L$ is dense in $Trans_L$.*

**Proof.** Let $D$ be an object of $Trans_L$. In order to show the density of $Tree_L$ in $Trans_L$ we have to show $D$ is the colimit of the canonical diagram

$$i(-)/D \longrightarrow Tree_L \longrightarrow Trans_L.$$

Fix a numbering $\alpha$ on $D$. Let $G : \Gamma(D)^{op} \longrightarrow i(-)/D$ be the functor defined by $G(x) = \gamma_x$ and $G((x,z)) = E((x,z))$ for $x\epsilon\Gamma(D)^{op}$ and an edge $(x,z)$ of $\Gamma(D)$. It follows from Lemma 5.2 that

$$colim(i(-)/D \to Tree_L \to Trans_L)$$

is isomorphic to

$$colim(\Gamma(D)^{op} \to i(-)/D \to Tree_L \to Trans_L).$$

Because the diagram $\Gamma(D)^{op} \to i(-)/D \to Tree_L \to Trans_L = i \circ E$, we have

$$colim(i(-)/D \to Tree_L \to Trans_L) \cong colim(i \circ E) \cong D$$

by Lemma 5.3. Since $D$ was arbitrary, we have shown that the functor $i : Tree_L \longrightarrow Trans_L$ is dense. $\square$

**Corollary 5.5** *$Trans_L$ is complete.*

Note that this does not imply that products are given by a simple construction: see [17] for an explicit description of them.

Finally, since $Trans_L$ is the category of coalgebras for an endofunctor on $Set$ and since it has a small dense subcategory, the forgetful functor $U : Trans_L \longrightarrow Set$ has a right adjoint by Theorem 2.1 and Proposition 3.11. It also reflects isomorphisms by Proposition 2.2. Moreover, it is routine to check that $P_\omega(L \times -)$ weakly preserves pullbacks. So, by the proof of Corollary 4.6, we have

**Theorem 5.6** *The category $Trans_L$ has a subobject classifier.*

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